Consequently, the true value of the concentration of the Polyox solution in the main experiments, based on our determination of the effective drag reduction in flow of the solution in pipes of different diameters, is

$$c = c_0 (c/c_0)_{S_0=20\%} = 10^{-4} \cdot 0.07 = 7 \cdot 10^{-6} \ \ \mathrm{g/cm^3}.$$

This result does not conflict with the expected concentration of the Polyox solution ($c < 8 \cdot 10^{-6}$ g/cm^3) when the inevitable material losses associated with the particular technique for preparation of the solution are taken into account.

NOTATION

d, pipe diameter; t, flow temperature; c, weight concentration of polymer solution; ν, kinematic viscosity of water; ν_p, kinematic viscosity of polymer solution; $\eta = \nu_p/\nu$, relative viscosity of polymer solution; ρ, density of water; τ_W, τ_p, tangential frictional stresses at the wall in pipe flows of water and polymer solution, respectively; τ_*, threshold tangential frictional stress at the wall; v_S, average velocity in terms of mass flow of liquid in the pipe; λ_W, λ_p, coefficients of fluid friction in pipe flows of water and polymer solution, respectively; Re, Reynolds number; c_0, weight concentration of polymer solution at t, °C, for 60% drag reduction; $S = (\tau_W - \tau_p)/\tau_W$, drag reduction at v_S = const for flow of polymer solution; S_0, maximum drag reduction for flow of a polymer solution of concentration c; c_0, characteristic concentration of polymer solution for maximum drag reduction $S_0 = 60\%$.

LITERATURE CITED

1. Yu. F. Ivanyuta and L. A. Chekalova, Inzh.-Fiz. Zh., 21, No. 1 (1971).
2. Yu. F. Ivanyuta and L. A. Chekalova, Inzh.-Fiz. Zh., 26, No. 5 (1974).
3. R. H. Forester, R. E. Larson, J. W. Haylen, and J. M. Wetzel, J. Hydronautics, 3, No. 1 (1969).
4. H. Schlichting, Boundary-Layer Theory, 6th ed., McGraw-Hill, New York (1968).
5. V. N. Kalashnikov et al., Zh. Prikl. Mekh. Tekh. Fiz., No. 6 (1968).

PERISTALTIC FLOW OF A NON-NEWTONIAN VISCOPLASTIC LIQUID IN A SLOT CHANNEL

V. I. Vishnyakov, K. B. Pavlov, and A. S. Romanov

UDC 532.54: 532.135

The "narrow-band" asymptotic method [5] has been used to consider the peristaltic flow of a viscoplastic medium in a slot channel. It is found that the mode of flow differs substantially from that in a channel with rigid walls when the axial pressure gradient is small.

Considerable attention has recently been given to the flow of liquids in channels with elastic walls in connection with many aspects of biomechanics [1], with particular interest attaching to non-Newtonian fluids with anomalous mechanical properties [2]. One class of non-Newtonian liquid is that of nonlinear-viscosity media, for which the simplest rheological law is one that relates the stress tensor deviator s_{ij} to the strain-rate tensor f_{ij}. In particular, a viscoplastic liquid is a medium with nonlinear viscosity, for which the rheological law can be put in the following form [3]:

$$\begin{gathered} s_{ij} = 2\left[\eta + \tau_0/(2f_{ij}f_{ij})^{1/2}\right] f_{ij} \qquad \text{for} \qquad (2s_{ij}s_{ij})^{1/2} \geqslant \tau_0, \\ f_{ij} = 0 \qquad \text{for} \qquad (2s_{ij}s_{ij})^{1/2} \leqslant \tau_0. \end{gathered} \tag{1}$$

Here we consider the peristaltic motion of a viscoplastic liquid (1) in a slot channel with elastic walls; in the general case, the peristaltic flow of the medium is due to the joint action of the deformable walls and a

N. É. Bauman Moscow Higher Technical School, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 31, No. 3, pp. 499-505, September, 1976. Original article submitted June 30, 1975.

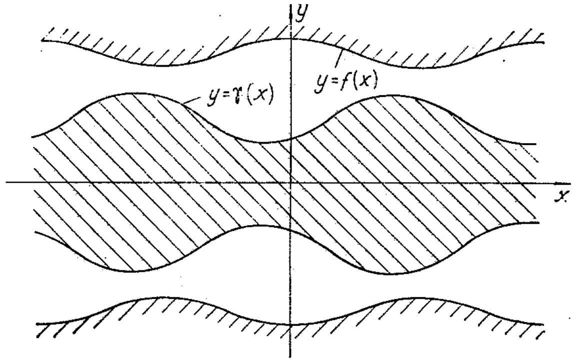


Fig. 1. Scheme for the peristaltic flow.

pressure gradient along the axis. In particular, such a flow is considered for a non-Newtonian liquid with power-law rheological behavior, which resembles a viscoplastic liquid in being a medium with nonlinear viscosity [4].

We assume that the deformation of the elastic wall is described by a traveling-wave equation having a wavelength much greater than the mean size of the channel; there exists a frame of reference moving along the axis x of the channel with a speed W relative to the wall in which the deformable wall is described by a time-independent equation $y = f(x)$. This frame will be called the "moving" frame, to distinguish it from the "immobile" one, in which the wall has no axial displacement. If the flow of the viscoplastic liquid in the "moving" system is independent of time, then the peristaltic motion will be said to be of steady-state type, and this is the only case considered here.

A distinctive feature of the steady-state peristaltic motion for a viscoplastic fluid is that there may be zones with different forms of analytical description for the velocity distribution: a region of viscous flow near the channel walls, $\gamma(x) \le y \le f(x)$, $-f(x) \le y \le -\gamma(x)$, and a quasisolid zone $-\gamma(x) \le y \le \gamma(x)$, which lies at the center (Fig. 1), with the speed in the quasisolid zone everywhere constant. If we use the characteristic quantities h — the mean half-width of the channel — and the speed W, then the equations of motion in the moving frame that describe the flow in the viscous zone are as follows in dimensionless form in terms of the stream function ψ ($d\psi/dy$ and $-d\psi/dx$ are, respectively, the x and y projections of the velocity):

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\frac{\partial\Delta\psi}{\partial y}$$
$$+\frac{\varkappa}{\mathrm{Re}}\left[2\frac{\partial}{\partial x}\omega^{-1}\frac{\partial^2\psi}{\partial x\partial y}+\frac{\partial}{\partial y}\omega^{-1}\left(\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right)\right]; \qquad (2)$$

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y}=\frac{\partial p}{\partial y}-\frac{1}{\mathrm{Re}}\frac{\partial\Delta\psi}{\partial x}$$
$$+\frac{\varkappa}{\mathrm{Re}}\left[2\frac{\partial}{\partial y}\omega^{-1}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial}{\partial x}\omega^{-1}\left(\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right)\right], \qquad (3)$$

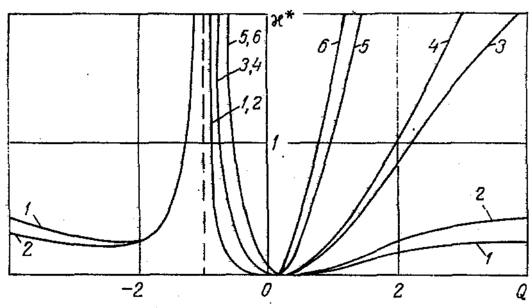
where

$$\omega=\left[4\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right)^2\right]^{1/2},$$

where $\mathrm{Re} = \rho W h/\eta$ is Reynolds number and $\varkappa = \tau_0 h/\eta W$ is the plasticity parameter. We eliminate the pressure p from (2) and (3) to get

$$\frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y}=\frac{\Delta\Delta\psi}{\mathrm{Re}}+\frac{\varkappa}{\mathrm{Re}}\left(4\frac{\partial^2}{\partial x\partial y}\omega^{-1}\frac{\partial^2\psi}{\partial x\partial y}+\frac{\partial^2\Omega}{\partial y^2}-\frac{\partial^2\Omega}{\partial x^2}\right);$$
$$\Omega=\left(\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right)\Big/\omega. \qquad (4)$$

The symmetry of the flow allows us to restrict consideration to the regions $y \ge 0$ and to write conditions that should be met by ψ at the boundaries of the viscous-flow zone $\gamma(x) \le y \le f(x)$ in the "moving" system:



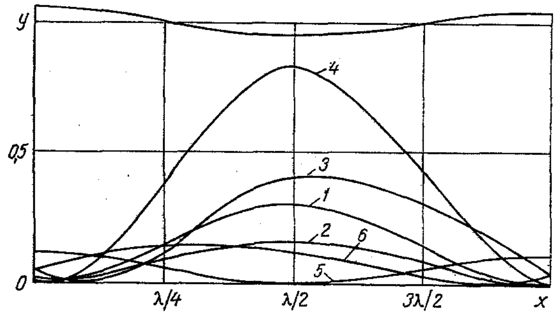


Fig. 2 **Fig. 3**

Fig. 2. Variation in the minimal value of the plasticity parameter $\varkappa^*$ with the flow rate Q for the following cases: $2\pi\mathrm{Re}/\lambda = 0$: 1) $a = 0.05$; 3) 0.15; 5) 0.25 and $2\pi\mathrm{Re}/\lambda = 0.1$: 2) $a = 0.05$; 4) 0.15; 6) 0.25.

Fig. 3. Waveforms arising at the surface of the quasisolid zone for the following values of the parameters: $a = 0.05$; $\varkappa = \varkappa^*$: $2\pi\mathrm{Re}/\lambda = 0$ for: 1) u = −3; 2) 1.5 and $2\pi\mathrm{Re}/\lambda = 0.3$ for: 3) u = −3; 4) 1.5; 5) 0; 6) 1.5.

$$\left.\frac{\partial\psi}{\partial y}\right|_{f(x)} = -1, \quad \left.\frac{\partial\psi}{\partial x}\right|_{f(x)} = \frac{df}{dx}, \tag{5}$$

$$\left.\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right)\right|_{\gamma(x)} = 0, \quad \left.\frac{\partial\psi}{\partial y}\right|_{\gamma(x)} = u, \tag{6}$$

$$\left.\frac{\partial\psi}{\partial x}\right|_{\gamma(x)} = 0. \tag{7}$$

Here u = U/W is the dimensionless velocity of the quasisolid zone [note that there exists also the boundary condition $(\partial^2\psi/\partial x\partial y)|_{\gamma(x)} = 0$, but this can be shown to follow from (6) and (7)].

The boundary-value problem of (4)-(6) may be solved by the "narrow-band" asymptotic method [5, 6]; we make the change of variable $x \to \varepsilon x$, in which ε is a small parameter that characterizes the narrowness of a "band," for instance, the ratio of the mean half-width of the channel to the wavelength of the deformation of the walls. Then this parameter ε appears in all the above equations and boundary conditions, which enables us to solve (4) as an asymptotic expansion with respect to ε:

$$\psi = \sum_{i=0}^{\infty} \varepsilon^i \psi_i; \qquad \gamma = \sum_{i=0}^{\infty} \varepsilon^i \gamma_i. \tag{8}$$

The zeroth approximation ψ_0 may be found for the viscous zone via the equation

$$\frac{\partial^4\psi_0}{\partial y^4} = 0, \tag{9}$$

which must be solved subject to the boundary conditions of (5) and (6), which take the following forms, respectively, in the zeroth approximation:

$$\begin{gathered} \left.\frac{\partial\psi_0}{\partial y}\right|_{f(x)} = -1, \quad \left.\frac{\partial\psi_0}{\partial x}\right|_{f(x)} = \frac{df}{dx}, \\ \left.\frac{\partial^2\psi_0}{\partial y^2}\right|_{\gamma(x)} = 0, \quad \left.\frac{\partial\psi_0}{\partial y}\right|_{\gamma(x)} = u. \end{gathered} \tag{10}$$

It is clear that the function $\psi_0 = C_1y^3 + C_2y^2 + C_3y + C_4$; $C_1 = -(u + 1)/3(\gamma_0 - f)^2$; $C_2 = -3C_1\gamma_0$; $C_3 = -1 - 3C_1 f(f - 2\gamma_0)$; $C_4 = -C_1f^3 - C_2f^2 - C_3f + \mathrm{const}$ is a solution to (9) that satisfies (10).

We use the following to determine the first approximation:

$$\frac{\partial^4\psi_1}{\partial y^4} = \mathrm{Re}\left(\frac{\partial\psi_0}{\partial y}\frac{\partial^3\psi_0}{\partial x\partial y^2} - \frac{\partial\psi_0}{\partial x}\frac{\partial^3\psi_0}{\partial y^3}\right);$$

$$\left.\frac{\partial\psi_1}{\partial y}\right|_{f(x)} = \left.\frac{\partial\psi_1}{\partial x}\right|_{f(x)} = \left.\frac{\partial^2\psi_1}{\partial y^2}\right|_{\gamma(x)} = \left.\frac{\partial\psi_1}{\partial y}\right|_{\gamma(x)} = 0;$$

$$\psi_1 = C_5y^3 + C_6y^2 + C_7y + C_8 + \mathrm{Re}\,S(x,\, y),$$

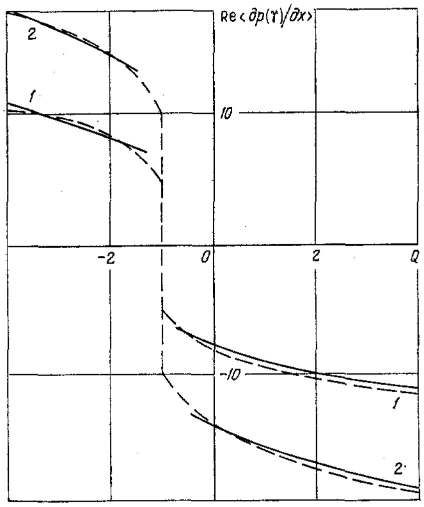


Fig. 4. Pressure gradient averaged over a period Re < ∂p/∂x > as a function of the flow rate Q for $2\pi \mathrm{Re}/\lambda = 0.1$; 1) $a = 0.05$, $\varkappa = 5$; 2) $a = 0.05$, $\varkappa = 10$; dashed line $a = 0$, $\varkappa = 5$ and 10.

where S (x, y) is a particular solution to the corresponding inhomogeneous equation, which is not given on account of its cumbersome form;

$$C_5 = -\mathrm{Re}\left[\frac{S''(x,\ \gamma_0)}{3(\gamma_0 - f)} + \frac{S'(x,\ f) - S'(x,\ \gamma_0)}{3(\gamma_0 - f)^2}\right] - \frac{2C_1\gamma_1}{(\gamma_0 - f)};$$

$$C_6 = -\frac{\mathrm{Re}}{2} S''(x,\ \gamma_0) - 3C_1\gamma_1 - 3C_5\lambda_0;$$

$$C_7 = \mathrm{Re}\,[fS''(x,\ \gamma_0) - S'(x,\ f)] - 3C_5 f(f - 2\gamma_0) + 6C_1\gamma_1 f;$$

$$C_8 = -\mathrm{Re}\,S(x,\ f) - x_5 f^3 - C_6 f^2 - C_7 f;$$

$$S'(x,\ f) \equiv \left.\frac{\partial S(x,\ y)}{\partial y}\right|_{f(x)}; \quad S'(x,\ \gamma_0) \equiv \left.\frac{\partial S(x,\ y)}{\partial y}\right|_{\gamma_0(x)};$$

$$S''(x,\ f) \equiv \left.\frac{\partial^2 S(x,\ y)}{\partial y^2}\right|_{f(x)}; \quad S''(x,\ \gamma_0) \equiv \left.\frac{\partial^2 S(x,\ y)}{\partial y^2}\right|_{\gamma_0(x)}.$$

From the remaining unused condition in (7) we get

$$\gamma_0 = \left(\frac{3}{u+1} - 2\right)\int_0^x df + \gamma_0(0);$$

$$\gamma_1 = \frac{3}{u+1}\int_0^x d\left\{\frac{1}{3} S''(x,\ \gamma_0)(\gamma_0 - f)^2 - \frac{1}{3}[S'(x,\ f) + 2S'(x,\ \gamma_0)](\gamma_0 - f) + S(x,\ \gamma_0) - S(x,\ f)\right\} + \gamma_1(0),$$

where $\gamma_0(0)$ and $\gamma_1(0)$ are constants of integration; we put $\gamma_0(0) = \gamma(0)$ to get $\gamma_1(0) = 0$, while the undetermined constant $\gamma_0(0)$ can be found from the equilibrium condition for the quasisolid zone, which when applied to the part of that zone between the sections x and x + 2π gives the following expression:

$$\Delta P\gamma(x) = \int_x^{x+2\pi} p(x,\ \gamma)\frac{d\gamma}{dx}\,dx + \frac{\varkappa 2\pi}{\mathrm{Re}}\,\mathrm{sign}\,\frac{\partial^2\psi}{\partial y^2}.$$

Here ΔP is the pressure difference in the quasisolid zone, while $p(x,\ \gamma)$ is the pressure in the viscous zone for $y = \gamma(x)$; it is clear that we can always state a value $x = x^*$ such that $\int_{x^*}^{x^*+2\pi} p(x,\ \gamma)\frac{d\gamma}{dx}\,dx = 0$, and we then suppose that the pressure differences between the x* and x* + 2π section are equal for the quasisolid zone and the viscous-flow one, which gives the desired relationship between $\gamma_0(0)$ and $\varkappa$ in the form

$$\varkappa = -\operatorname{sign}\frac{\partial^2\psi}{\partial y^2}\,\frac{\gamma^*}{2\pi}\int\limits_0^{2\pi}\frac{u+1}{3}\,\frac{dx}{(\gamma_0-f)^2},$$

where the value $\gamma^* = \gamma(x^*)$ is defined by the condition

$$\int\limits_{x^*}^{x^*+2\pi}\frac{d\gamma}{dx}\,dx\int\limits_{x^*}^{x}\frac{dx}{(\gamma_0-f)^2} = 0.$$

This asymptotic solution was used in numerical calculations for the particular case where the deformation of the elastic walls is described by

$$f(x) = 1 + a\cos 2\pi x/\lambda \qquad (0 < a < 1).$$

The results showed that there is a minimum value for the plasticity parameter $\varkappa^*$ such that $\gamma(x)$ becomes zero at certain points in the channel; i. e., the quasisolid zone "breaks up" for specified values of a, Re, λ, and the dimensionless flow rate Q; Fig. 2 shows $\varkappa^*$ in relation to the definitive parameters.

Figure 3 shows the style of the waves at the surface of the quasisolid zone in relation to the velocity u for the case $\varkappa = \varkappa^*$; even small deformations of the elastic wall can produce states of flow in the viscoplastic medium such that the wave amplitudes at the surface of the quasisolid zone are close to the characteristic size of the channel (curve 4 in Fig. 3).

Figure 4 shows the pressure gradient averaged over a period Re $\langle \partial p(\gamma)/\partial x\rangle$ in relation to the flow rate Q and the plasticity parameter $\varkappa$ for $a = 0.05$ and $2\pi\,\mathrm{Re}/\lambda = 0.1$; if $\langle \partial p/\partial x\rangle < 0$, the medium flows in the positive direction of the x axis, while for $\langle \partial p/\partial x\rangle > 0$ it flows in the opposite sense. The dashed line in Fig. 4 corresponds to flow in a channel with rigid walls. A viscoplastic medium can flow in such a channel only if $\mathrm{Re}|\partial p/\partial x| > \varkappa$, whereas in the case of a channel with elastic walls there is no such restriction. The discontinuity in the region Q = −1 is due to a "breakup" of the quasisolid zone and formation of a mode of flow essentially different from that considered above.

NOTATION

x, y, Cartesian coordinates; τ_0, yield stress; ρ, density; η, dynamic viscosity; W, phase velocity of deformation wave; h, mean channel half-width; ψ, stream function; Re, Reynolds number; $\varkappa$, plasticity parameter; s_{ij}, stress-tensor deviator; f_{ij}, strain-rate tensor; U, quasisolid zone velocity; u, dimensionless quasisolid zone velocity; a, amplitude; λ, wavelength; p, pressure; ε, small parameter; Q, flow rate.

LITERATURE CITED

1. S. A. Regirer and I. M. Skobeleva, "Flow of a viscous liquid in a porous tube with a deformable wall," Izv. Akad. Nauk SSSR, No. 3, 118 (1971).
2. Yu. N. Pavlovskii, S. A. Regirer, and I. M. Skobeleva, Hydrodynamics of Blood: Surveys of Science, 1968, Hydromechanics [in Russian], VINITI, Moscow (1970).
3. M. Reiner, Lectures on Theoretical Rheology, 3rd ed., Humanities (1960).
4. V. I. Vishnyakov and K. B. Pavlov, Inzh.-Fiz. Zh., 26, No. 2, 246 (1974).
5. Yu. P. Ivanilov, N. N. Moiseev, and A. M. Ter-Krikorov, Dokl. Akad. Nauk SSSR, 123, No. 2, 231 (1953).
6. N. N. Moiseev, in: Some Problems in Mathematics and Mechanics [in Russian], Izd. Sibirsk. Otd., Akad. Nauk SSSR, Novosibirsk (1961).